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A Criterion for Self-Adjointness of Singular Elliptic Differential Operators

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Let G be an open r-times connected set in R_n, $[x = (x_1, x_2, \dots, x_n)]$. Especially, G need not be bounded and we also admit $G = R_n$.

Let $\mathfrak{H}$ be the Hilbert space

$$\mathfrak{H} = \left\{ u(x) \,\Big|\, \int_G |u(x)|^2 k(x)\, dx < \infty \right\}, \qquad (k(x) > 0)$$

with scalar product

$$(u, v) = \int_G u\bar{v}k\, dx.$$

We consider the operator A in $\mathfrak{A}$ with

$$Au = \frac{1}{k(x)} \left\{ \sum_{j,l=1}^{n} D_j[p_{jl}(x)\, D_l u(x)] + q(x)\, u(x) \right\} \tag{1}$$

$$\left(\text{where } D_j u(x) \equiv \left[i \frac{\partial}{\partial x_j} + b_j(x) \right] u(x) = iu_{x_j} + b_j(x)\, u \right)$$

and

$$\mathfrak{A} = \{u(x) \mid u \in C^2(G) \cap \mathfrak{H}, Au \in \mathfrak{H}\}.$$

We make the following assumptions:

(1) There exists in $G - B$ (where B is a compact subset of G) a real-valued function $\varphi(x) \in C^1(G - B)$ with $\varphi(x) > 0$, $|\operatorname{grad} \varphi| > 0$, such that for every sequence of points in G $x^{(1)}, x^{(2)}, \dots, x^{(m)}$, for which the distance of the points to a boundary point of G tends to 0, when $m \to \infty$, or for which $\lim_{m\to\infty} |x^{(m)}| = \infty$ holds, it is $\lim_{m\to\infty} \varphi(x^{(m)}) = 0$. Setting

$$G_t = G - \{x \mid x \in G - B, \varphi(x) \leqslant t\},$$

we conclude that $G_{t_1} \supset G_{t_2}$ for $0 < t_1 < t_2$. The boundary of G_t is, for sufficiently small t, (say $t \leqslant t_0$)

$$\partial G_t = \{x \mid \varphi(x) = t\}.$$

Finally it is in an obvious notation:

$$\lim_{t \to 0} G_t = G.$$

(2) $p_{jl}(x)$, $b_j(x)$, $q(x)$ real-valued.

$p_{jl}(x) = p_{lj}(x)$; $p_{jl}(x)$ twice Hölder-continuously differentiable in G, $q(x)$, $k(x)$ Hölder-continuous in G.

$$\sum_{j,l=1}^{n} p_{jl}(x)\, \xi_j \bar{\xi}_l \geqslant \mu(x) \sum_{j=1}^{n} |\xi_j|^2 \tag{3}$$

for arbitrary complex numbers $\xi_1, \xi_2, \ldots, \xi_n$. $\mu(x)$ is real-valued and $\mu(x) > 0$ for $x \in G$.

We further consider A in the subspaces

$$\mathring{\mathfrak{A}} = \{u(x) \mid u \in C^2(G), u \text{ has compact support}\}$$

and

$$\mathring{\mathfrak{C}} = \{u(x) \mid u \in C^\infty(G), u \text{ has compact support}\}.$$

A theorem of Wienholtz [13] states that A in $\mathring{\mathfrak{C}}$ is essentially self-adjoint if A in $\mathfrak{A}$ is symmetric.[1] Then A in $\mathring{\mathfrak{A}}$ is a fortiori essentially self-adjoint and the closure $\bar{A}$ in $\bar{\mathfrak{A}}$ is self-adjoint. With [8] it follows the additional result that all eigenfunctions and eigenpackets of the self-adjoint operator $\bar{A}$ in $\bar{\mathfrak{A}}$ are even contained in $\mathfrak{A}$ and hence the eigenfunctions and eigenpackets form a complete system in $\mathfrak{A}$.

[1] We use the following notations:

(α) T in $\mathfrak{T}$ is symmetric if $\mathfrak{T}$ is dense in $\mathfrak{H}$ and $(Tu, v) = (u, Tv)$ holds for all $u, v \in \mathfrak{T}$.

(β) T in $\mathfrak{T}$ is essentially self-adjoint if T in $\mathfrak{T}$ is symmetric and the spaces $(T + iE)\mathfrak{T}$, $(T - iE)\mathfrak{T}$ are dense in $\mathfrak{H}$.

(γ) T in $\mathfrak{T}$ is self-adjoint if T in $\mathfrak{T}$ is symmetric and $(T + iE)\mathfrak{T} = \mathfrak{H}$, $(T - iE)\mathfrak{T} = \mathfrak{H}$. This definition is equivalent to $T = T^*$.

If T in $\mathfrak{T}$ is essentially self-adjoint, then the closure T in $\mathfrak{T}$ is self-adjoint and hence has a spectral decomposition. For a more detailed presentation see G. Hellwig [3].

In this paper we will give a criterion for the symmetry of the operator A in $\mathfrak{A}$ and with the theorem of Wienholtz [13], we then can conclude that $\bar{A}$ in $\bar{\mathfrak{A}}$ is self-adjoint. The criterion given here is a generalization of a criterion given in [2]. Other criteria for self-adjointness can be found in Ikebe-Kato [4], Jörgens [5], Rohde [7], [9], Stummel [10], Walter [11], Wienholtz [12].

THEOREM 1. *In addition to the assumptions* (1), (2), (3), *we make the following assumption:*

Putting

$$\rho(t) = \sup_{\varphi(x)=t} \frac{1}{k(x)} \sum_{j,l=1}^{n} p_{jl}(x)\, \varphi_{x_j}(x)\, \varphi_{x_l}(x)$$

for $0 < t < t_0$, *there exists a positive function* $M(t) \in C^1$ $(0 < t \leqslant t_0)$ *with*

$$\lim_{t\to 0} \int_t^{t_0} \frac{d\tau}{\sqrt{\rho(\tau)\, M(\tau)}} = \infty \tag{2}$$

and

$$\frac{1}{M(t)} \leqslant C_1 \left[\int_t^{t_0} \frac{d\tau}{\sqrt{\rho(\tau)\, M(\tau)}} \right]^2, \tag{3}$$

$$\frac{1}{k(x)} \left\{ \frac{2}{\epsilon} \sum_{j,l=1}^{n} p_{jl}(x)\, \varphi_{x_j}(x)\, \varphi_{x_l}(x) \left[\frac{d}{dt} \left(\frac{1}{\sqrt{M(\varphi(x))}} \right) \right]^2 - \frac{q(x)}{M(\varphi(x))} \right\}$$
$$\leqslant C_2 \left[\int_t^{t_0} \frac{d\tau}{\sqrt{\rho(\tau)\, M(\tau)}} \right]^2 \tag{4}$$

for sufficiently small t with $\varphi(x) = t$, where $C_1 > 0$, $C_2 > 0$, $0 < \epsilon < 2$ are constants. Then A in $\mathfrak{A}$ is symmetric.

REMARK. We note that (2), (3), and (4) do not contain any conditions on $b_j(x)$.

PROOF OF THE THEOREM. We first state two lemmas.

LEMMA 1 (Schwarz inequality for quadratic forms). *If α_j, β_j, $j = 1, 2,..., n$ are complex numbers, then it is*

$$\left| \sum_{j,l=1}^{n} p_{jl}(x)\, \alpha_j \bar{\beta}_l \right|^2 \leqslant \sum_{j,l=1}^{n} p_{jl}(x)\, \alpha_j \bar{\alpha}_l \sum_{j,l=1}^{n} p_{jl}(x)\, \beta_j \bar{\beta}_l\,. \tag{5}$$

Lemma 2. *For* $0 < \tilde{t} < t_0$ *there exists a sequence* t_m, *with* $\lim_{m\to\infty} t_m = 0$ *and a constant* C_∞ *(depending on* u *and* $\tilde{t}$*) such that for all* $m > m_0$ *it is*

$$\int_{G_{t_m}-G_{\tilde{t}}} \frac{\sum_{j,l=1}^{n} p_{jl}(x)\, D_j u(x)\, \overline{D_l u(x)}}{M(\varphi(x))}\, dx \leqslant C_\infty \left[\int_{t_m}^{\tilde{t}} \frac{d\tau}{\sqrt{\rho(\tau) M(\tau)}}\right]^2. \tag{6}$$

Proof of Lemma 2. Because of (2), we can, without loss of generality, assume that $\tilde{t}$ is so small that (3) and (4) holds for $t < \tilde{t}$ and it is

$$\int_{G-G_{\tilde{t}}} |\, u \,|\, |\, Au \,|\, k\, dx < \frac{1}{12C_1}, \tag{7}$$

$$\int_{G-G_{\tilde{t}}} |\, u \,|^2\, k\, dx < inf \left[\frac{1}{12C_1}, \frac{d}{36C_1^2}\right], \tag{8}$$

where

$$d = 1 - \frac{\epsilon}{2}.$$

If Lemma 2 were not true, then

$$\lim_{t\to 0} \frac{\displaystyle\int_{G_t-G_{\tilde{t}}} \frac{\sum p_{jl}\, D_j u\, \overline{D_l u}}{M(\varphi)}\, dx}{\left[\displaystyle\int_t^{\tilde{t}} \frac{d\tau}{\sqrt{\rho M}}\right]^2} = \infty. \tag{9}$$

We set

$$F(t) = \int_{G_t-G_{\tilde{t}}} \frac{\sum p_{jl}\, D_j u\, \overline{D_l u}}{M(\varphi)}\, dx. \tag{10}$$

Then $F(t)$ is monotonically increasing for decreasing t. With (2) and our assumption to the contrary (9), it is

$$\lim_{t\to 0} F(t) = \infty.$$

We consider

$$\int_{G_t-G_{\tilde{t}}} \frac{u\overline{Au}}{M(\varphi)}\, k\, dx = \int_{G_t-G_{\tilde{t}}} \frac{u}{M(\varphi)} \overline{\left[- \sum D_j(p_{jl}\, D_l u) - qu\right]}\, dx.$$

Recalling that we have set

$$D_j u = i u_{x_j} + b_j(x)\, u,$$

we obtain

$$\int_{G_t - G_{\tilde t}} \left\{ \frac{u}{M(\varphi)} \sum [i(p_{jl}\, D_l u)_{x_j} - b_j p_{jl}\, \overline{D_l u}] - \frac{q\,|\,u\,|^2}{M(\varphi)} \right\} dx$$

$$= \int_{G_t - G_{\tilde t}} \left\{ \sum \left[i \left(\frac{u}{M(\varphi)}\, p_{jl}\, \overline{D_l u} \right)_{x_j} - i \left(\frac{u}{M(\varphi)} \right)_{x_j} p_{jl}\, \overline{D_l u} \right.\right.$$

$$\left.\left. - \frac{u}{M(\varphi)}\, b_j p_{jl}\, \overline{D_l u} \right] - \frac{q\,|\,u\,|^2}{M} \right\} dx. \tag{11}$$

We set

$$f(t) = \int_{\partial G_t} \sum i\, \frac{u}{M(\varphi)}\, p_{jl} \nu_j\, \overline{D_l u}\, dS, \tag{12}$$

where $\nu = (\nu_1, \nu_2, ..., \nu_n)$ is the outer normal of ∂G_t.

We apply the Gauss integral theorem on $G_t - G_{\tilde t}$ and obtain with $\varphi(x) = \tau$ on ∂G_τ

$$- \int_{G_t - G_{\tilde t}} \frac{u}{M(\varphi)}\, \overline{A u} k\, dx$$

$$= f(t) - f(\tilde t) - \int_{G_t - G_{\tilde t}} \frac{[iu_{x_j} + b_j]\, p_{jl}\, \overline{D_l u}}{M(\varphi)}\, dx$$

$$+ i \int_{G_t - G_{\tilde t}} \sum \frac{u}{[M(\varphi)]^2} \frac{d}{d\tau} [M(\varphi)]\, \varphi_{x_j} p_{jl}\, \overline{D_l u}\, dx - \int_{G_t - G_{\tilde t}} \frac{q\,|\,u\,|^2}{M(\varphi)}\, dx. \tag{13}$$

With notations (10) and (12) we get from (13)

$$F(t) = f(t) - f(\tilde t) + \int_{G_t - G_{\tilde t}} \frac{u}{M(\varphi)}\, \overline{A u} k\, dx$$

$$+ i \int_{G_t - G_{\tilde t}} \sum \frac{u}{[M(\varphi)]^2} \frac{d}{d\tau} [M(\varphi)]\, \varphi_{x_j} p_{jl}\, \overline{D_l u}\, dx - \int_{G_t - G_{\tilde t}} \frac{q\,|\,u\,|^2}{M(\varphi)}\, dx$$

and hence

$$2F(t) = f(t) + \overline{f(t)} - [f(\tilde t) + \overline{f(\tilde t)}] + \int_{G_t - G_{\tilde t}} \left(\frac{u}{M(\varphi)}\, \overline{Au} + \frac{\bar u}{M(\varphi)}\, Au \right) k\, dx$$

$$+ i \int_{G_t - G_{\tilde t}} \frac{\frac{d}{d\tau} [M(\varphi)]}{[M(\varphi)]^2}\, \nu_j p_{jl} (u\, \overline{D_l u} - \bar u\, D_l u)\, dx$$

$$- 2 \int_{G_t - G_{\tilde t}} \frac{q\,|\,u\,|^2}{M(\varphi)}\, dx.$$

With (3) and (7) it is

$$\begin{aligned}
&\left| \int_{G_t - G_{\tilde t}} \left(\frac{u}{M(\varphi)} \overline{Au} + \frac{\bar u}{M(\varphi)} Au \right) k \, dx \right| \\
&\leqslant 2 \int_{G_t - G_{\tilde t}} \frac{1}{M(\varphi)} \mid u \mid \mid Au \mid k \, dx \\
&\leqslant 2C_1 \left[\int_t^{t_0} \frac{d\tau}{\sqrt{\rho(\tau)\, M(\tau)}} \right]^2 \int_{G_t - G_{\tilde t}} \mid u \mid \mid Au \mid k \, dx \\
&\leqslant \frac{2C_1}{12 C_1} \left[\int_t^{t_0} \frac{d\tau}{\sqrt{\rho M}} \right]^2 = \frac{1}{6} \left[\int_t^{t_0} \frac{d\tau}{\sqrt{\rho M}} \right]^2 .
\end{aligned} \tag{14}$$

Further, with Lemma 1, it is for $0 < \epsilon < 2$ and $\varphi(x) = \tau$

$$\begin{aligned}
&\left| i \int_{G_t - G_{\tilde t}} \frac{\frac{d}{d\tau} M(\varphi)}{[M(\varphi)]^2} \varphi_{x_j} p_{jl} (u \overline{D_l u} - \bar u \, D_l u) \, dx \right| \\
&\leqslant 2 \int_{G_t - G_{\tilde t}} \left[\frac{\sum p_{jl} \, D_j u \, \overline{D_l u}}{M(\varphi)} \right]^{1/2} \left[\frac{\sum p_{jl} \varphi_{x_j} \varphi_{x_l} \mid u \mid^2 \left| \frac{d}{d\tau} M(\varphi) \right|^2}{[M(\varphi)]^3} \, dx \right]^{1/2} \\
&\leqslant \int_{G_t - G_{\tilde t}} \left[\frac{\epsilon \sum p_{jl} \, D_j u \, \overline{D_l u}}{M(\varphi)} + \frac{1}{\epsilon} \sum p_{jl} \varphi_{x_j} \varphi_{x_l} \mid u \mid^2 \frac{\left| \frac{d}{d\tau} M(\varphi) \right|^2}{(M(\varphi))^3} \right] dx \\
&= \epsilon F(t) + \frac{4}{\epsilon} \int_{G_t - G_{\tilde t}} \sum p_{jl} \varphi_{x_j} \varphi_{x_l} \mid u \mid^2 \left| \frac{d}{d\tau} \left(\frac{1}{\sqrt{M(\varphi)}} \right) \right|^2 dx.
\end{aligned}$$

Thus, with (4), we get

$$\begin{aligned}
F(t) &\leqslant \tfrac{1}{2} [f(t) + \overline{f(t)}] - \tfrac{1}{2} [f(\tilde t) + \overline{f(\tilde t)}] \\
&\quad + \frac{1}{12} \left[\int_t^{t_0} \frac{d\tau}{\sqrt{\rho M}} \right]^2 + \frac{\epsilon}{2} F(t) \\
&\quad + \frac{2}{\epsilon} \int_{G_t - G_{\tilde t}} \frac{\sum p_{jl} \varphi_{x_j} \varphi_{x_l}}{k} \left| \frac{d}{d\tau} \left(\frac{1}{\sqrt{M(\varphi)}} \right) \right|^2 \mid u \mid^2 k \, dx \\
&\quad - \int_{G_t - G_{\tilde t}} \frac{q \mid u \mid^2}{M(\varphi)} \, dx \\
&\leqslant \tfrac{1}{2} [f(t) + \overline{f(t)}] - \tfrac{1}{2} [f(\tilde t) + \overline{f(\tilde t)}] \\
&\quad + \frac{1}{12} \left[\int_t^{t_0} \frac{d\tau}{\sqrt{\rho M}} \right]^2 + \frac{\epsilon}{2} F(t) + C_2 \left[\int_t^{t_0} \frac{d\tau}{\sqrt{\rho M}} \right]^2 \int_{G_t - G_{\tilde t}} \mid u \mid^2 k \, dx.
\end{aligned}$$

For sufficiently small t, say for $t < t_* < \bar{t} < t_0$, it is because of (2)

$$\frac{1}{2}\,|f(\bar{t}) + \overline{f(\bar{t})}\,| < \frac{1}{12}\left[\int_t^{t_0} \frac{d\tau}{\sqrt{\rho M}}\right]^2 \tag{15}$$

and

$$\left[\int_{\bar{t}}^{t_0} \frac{d\tau}{\sqrt{\rho M}}\right]^2 < \left[\int_t^{\bar{t}} \frac{d\tau}{\sqrt{\rho M}}\right]^2. \tag{16}$$

Hence, with (15), (16), and (8), it is for $t < t_*$

$$\begin{aligned}
\left(1 - \frac{\epsilon}{2}\right) F(t) &\leqslant \frac{1}{2}\,[f(t) + \overline{f(t)}] + \frac{1}{12}\left[\int_t^{t_0} \frac{d\tau}{\sqrt{\rho M}}\right]^2 \\
&\qquad + \frac{1}{12}\left[\int_t^{t_0} \frac{d\tau}{\sqrt{\rho M}}\right]^2 + \frac{1}{12}\left[\int_t^{t_0} \frac{d\tau}{\sqrt{\rho M}}\right]^2 \\
&\leqslant \frac{1}{2}\,[f(t) + \overline{f(t)}] + \frac{1}{4}\left[\int_t^{t_0} \frac{d\tau}{\sqrt{\rho M}}\right]^2 \\
&\leqslant |f(t)| + \frac{1}{4}\left[2\left(\int_{\bar{t}}^{t_0} \frac{d\tau}{\sqrt{\rho M}}\right)^2 + 2\left(\int_t^{\bar{t}} \frac{d\tau}{\sqrt{\rho M}}\right)^2\right] \\
&\leqslant |f(t)| + \left[\int_t^{\bar{t}} \frac{d\tau}{\sqrt{\rho M}}\right]^2.
\end{aligned} \tag{17}$$

Further, with $0 < d = 1 - (\epsilon/2)$ and (9), it is for sufficiently small t, say $t < t_{**} < t_*$

$$F(t) \geqslant \frac{2}{d}\left[\int_t^{\bar{t}} \frac{d\tau}{\sqrt{\rho M}}\right]^2. \tag{18}$$

Hence, for $t < t_{**}$, we get from (17) and (18)

$$\left(1 - \frac{\epsilon}{2}\right) F(t) \leqslant |f(t)| + \frac{d}{2} F(t)$$

$$\frac{d}{2} F(t) \leqslant |f(t)|. \tag{19}$$

Dividing both sides of (19) by $\sqrt{\rho(t)\, M(t)}$ and integrating between t and t_{**} ($t < t_{**}$) we obtain with (5), (3), and (8)

$$\begin{aligned}
\frac{d}{2}\int_t^{t_{**}} \frac{F(\tau)}{\sqrt{\rho(\tau)\, M(\tau)}}\, d\tau &\leqslant \int_t^{t_{**}} \frac{1}{\sqrt{\rho(\tau)\, M(\tau)}} \left| \int_{\partial G_\tau} \sum \frac{iu}{M(\varphi)}\, p_{jl}\nu_j\, \overline{D_l u}\, dS \right| d\tau \\
&\leqslant \int_t^{t_{**}} \int_{\partial G_\tau} \left| \frac{1}{\sqrt{\rho(\varphi) M(\varphi)}} \sum \frac{u}{M(\varphi)}\, p_{jl}\nu_j\, \overline{D_l u} \right| dS\, d\tau \\
&\leqslant \int_t^{t_{**}} \int_{\partial G_\tau} C_1 \left[\int_\tau^{t_0} \frac{d\zeta}{\sqrt{\rho(\zeta)\, M(\zeta)}} \right]^2 \\
&\quad \times \left[\frac{\sum p_{jl}\varphi_{x_j}\varphi_{x_l} \mid u \mid^2 k}{\rho(\varphi)\, k} \right]^{1/2} \\
&\quad \times \left[\frac{\sum p_{jl}\, D_j u\, \overline{D_l u}}{M(\varphi)} \right]^{1/2} \frac{dS}{\mid \operatorname{grad} \varphi \mid}\, d\tau \\
&\leqslant C_1 \left[\int_t^{t_0} \frac{d\tau}{\sqrt{\rho(\tau)\, M(\tau)}} \right]^2 \left[\int_{G_t - G_{t_{**}}} \mid u \mid^2 k\, dx \right]^{1/2} \\
&\quad \times \left[\int_{G_t - G_{t_{**}}} \frac{\sum p_{jl}\, D_j u\, \overline{D_l u}}{M(\varphi)}\, dx \right]^{1/2} \\
&\leqslant C_1 \left[\int_t^{t_0} \frac{d\tau}{\sqrt{\rho(\tau)\, M(\tau)}} \right]^2 \frac{\sqrt{d}}{6C_1} [F(t)]^{1/2} \\
&= \frac{\sqrt{d}}{6} [F(t)]^{1/2} \left[\int_t^{t_0} \frac{d\tau}{\sqrt{\rho(\tau)\, M(\tau)}} \right]^2 .
\end{aligned} \tag{20}$$

Set

$$G(t) = \int_t^{t_{**}} \frac{F(\tau)}{\sqrt{\rho(\tau)\, M(\tau)}}\, d\tau. \tag{21}$$

Then

$$G'(t) = - \frac{F(t)}{\sqrt{\rho(t)\, M(t)}}. \tag{22}$$

Hence, (20) can be written as

$$\frac{d^2}{4}[G(t)]^2 \leqslant \frac{d}{36}(-1)\,G'(t)\sqrt{\rho(t)\,M(t)}\left[\int_t^{t_0}\frac{d\tau}{\sqrt{\rho(\tau)\,M(\tau)}}\right]^4$$

$$-\frac{G'(t)}{[G(t)]^2} \geqslant 9d\,\frac{1}{\sqrt{\rho(t)\,M(t)}}\,\frac{1}{\left[\int_t^{t_0}\frac{d\tau}{\sqrt{\rho(\tau)\,M(\tau)}}\right]^4}.$$

Integration between $\check{t}$ and t with $\check{t} < t < t_{**}$ gives

$$\frac{1}{G(t)} - \frac{1}{G(\check{t})} \geqslant \frac{9d}{3}\left\{\frac{1}{\left[\int_t^{t_0}\frac{d\tau}{\sqrt{\rho M}}\right]^3} - \frac{1}{\left[\int_{\check{t}}^{t_0}\frac{d\tau}{\sqrt{\rho M}}\right]^3}\right\}. \tag{23}$$

It is

$$G(t) = \int_t^{t_{**}}\frac{F(\tau)}{\sqrt{\rho(\tau)\,M(\tau)}}\,d\tau \geqslant F(t_{**})\int_t^{t_{**}}\frac{d\tau}{\sqrt{\rho(\tau)\,M(\tau)}} \to \infty,$$

when $t \to 0$. Hence, for $\check{t} \to 0$ we get from (23)

$$\frac{1}{G(t)} \geqslant \frac{3d}{\left[\int_t^{t_0}\frac{d\tau}{\sqrt{\rho(\tau)\,M(\tau)}}\right]^3}. \tag{24}$$

From (18), (21), and (24) we get

$$\int_t^{t_{**}}\frac{2}{d\sqrt{\rho(\tau)\,M(\tau)}}\left[\int_\tau^{\tilde{t}}\frac{d\zeta}{\sqrt{\rho(\zeta)\,M(\zeta)}}\right]^2 d\tau \leqslant \int_t^{t_{**}}\frac{F(\tau)}{\sqrt{\rho(\tau)\,M(\tau)}}\,d\tau$$

$$\leqslant \frac{1}{3d}\left[\int_t^{t_0}\frac{d\tau}{\sqrt{\rho(\tau)\,M(\tau)}}\right]^3.$$

Hence

$$-\frac{2}{3}\left[\int_{t_{**}}^{\tilde{t}}\frac{d\tau}{\sqrt{\rho M}}\right]^3 + \frac{2}{3}\left[\int_t^{\tilde{t}}\frac{d\tau}{\sqrt{\rho M}}\right]^3 \leqslant \frac{1}{3}\left[\int_t^{t_0}\frac{d\tau}{\sqrt{\rho M}}\right]^3,$$

$$-\left[\int_{t_{**}}^{\tilde{t}}\frac{d\tau}{\sqrt{\rho M}}\right]^3 + \left[\int_t^{\tilde{t}}\frac{d\tau}{\sqrt{\rho M}}\right]^3 \leqslant \frac{1}{2}\left[\int_t^{t_0}\frac{d\tau}{\sqrt{\rho M}}\right]^3.$$

$t \to 0$ gives a contradiction.

PROOF OF THE THEOREM. It is $\mathfrak{A}$ dense in $\mathfrak{H}$. In order to prove that $(Au, v) = (u, Av)$ for all $u, v \in \mathfrak{A}$, it is sufficient to prove $(Au, u) = (u, Au)$ for all $u \in \mathfrak{A}$ (see e.g. [3]).

Let u be an arbitrary function in $\mathfrak{A}$. We set

$$\psi(t) = \int_{G_t} Au\bar{u}k\,dx - \int_{G_t} \overline{Au}uk\,dx. \tag{25}$$

Because of $Au \in \mathfrak{H}$, we can conclude with the Schwarz inequality that

$$\lim_{t\to 0} |\psi(t)| = \psi_0$$

exists. We have to prove that $\psi_0 = 0$.

Assume to the contrary that $\psi_0 > 0$. Now, we choose a $\tilde{t}$ so small that

(1) $|\psi(t)| > \psi_0/2$ for all $t < \tilde{t}$;

(2) Lemma 2 holds, i.e.,

$$\int_{G_{t_m}-G_{\tilde{t}}} \frac{\sum p_{jl}\, D_j u\, \overline{D_l u}}{M(\varphi)}\, dx < C_\infty \left[\int_{t_m}^{\tilde{t}} \frac{d\tau}{\sqrt{\rho M}}\right]^2. \tag{26}$$

The constant C_∞ and the sequence $\{t_m\}$ depend on u and $\tilde{t}$. Having chosen such a fixed $\tilde{t}$, we now choose a $\hat{t} < \tilde{t}$ so small that

$$\int_{G-G_{\hat{t}}} |u|^2 k\,dx < \frac{\psi_0}{8C_\infty}. \tag{27}$$

With the Gauss integral theorem we get from (25)

$$\psi(t) = i\int_{\partial G_t} \sum_{j,l=1}^{n} [\bar{u}p_{jl}\nu_j\, D_l u + u p_{jl}\nu_j\, \overline{D_l u}]\, dS.$$

Integration between $\hat{t}$ and t_m gives with (27)

$$\begin{aligned}
\frac{\psi_0}{2}\left[\int_{t_m}^{\hat{t}} \frac{d\tau}{\sqrt{\rho M}}\right]^2 &\leqslant \left[\int_{t_m}^{\hat{t}} \frac{|\psi(\tau)|\, d\tau}{\sqrt{\rho(\tau)\, M(\tau)}}\right]^2 \\
&\leqslant 2\left[\int_{t_m}^{\hat{t}}\int_{\partial G_\tau} \left|\frac{\sum_{j,l=1}^{n} u p_{jl}\varphi_{x_j}\, \overline{D_l u}}{\sqrt{\rho(\varphi)\, M(\varphi)}}\right| \frac{dS}{|\operatorname{grad}\varphi|}\, d\tau\right]^2 \\
&\leqslant 2\left(\int_{t_m}^{\hat{t}} \left[\int_{\partial G_\tau} \frac{\sum_{j,l=1}^{n} p_{jl}\varphi_{x_j}\varphi_{x_l}\, |u|^2}{\rho(\varphi)\, k}\, k\, \frac{dS}{|\operatorname{grad}\varphi|}\right]^{1/2}\right. \\
&\qquad \left.\times \left[\int_{\partial G_\tau} \frac{\sum_{j,l=1}^{n} p_{jl}\, D_j u\, \overline{D_l u}}{M(\varphi)}\, \frac{dS}{|\operatorname{grad}\varphi|}\right]^{1/2} d\tau\right)^2 \\
&\leqslant 2\int_{G_{t_m}-G_{\hat{t}}} |u|^2 k\, dx\, F(t_m) \leqslant \frac{2\psi_0 C_\infty}{8C_\infty}\left[\int_{t_m}^{\tilde{t}} \frac{d\tau}{\sqrt{\rho M}}\right]^2.
\end{aligned}$$

Hence

$$1 \leqslant \tfrac{1}{2} \frac{\left[\int_{t_m}^{\tilde{t}} \frac{d\tau}{\sqrt{\rho M}}\right]^2}{\left[\int_{t_m}^{t} \frac{d\tau}{\sqrt{\rho M}}\right]^2} \to \tfrac{1}{2}, \qquad \text{when} \qquad m \to \infty, t_m \to 0.$$

Contradiction!

EXAMPLE. Let

$$Au = -\sum_{j=1}^{n} (e^{-|x|^2} u_{x_j})_{x_j} + q(x)\, u,$$

with

$$k(x) \equiv 1, \qquad G = R_n .$$

We choose

$$\varphi(x) = \frac{1}{|x|} = t.$$

Then it is

$$\rho(t) = e^{-(1/t^2)} t^4.$$

We choose

$$M(t) = \frac{e^{(1/t^2)}}{t^2} .$$

Then we get

$$\int_t^{t_0} \frac{d\tau}{\sqrt{\rho M}} = \log \frac{t_0}{t} .$$

Further it is

$$\frac{1}{M(t)} \to 0, \qquad \sum_{j,l=1}^{n} p_{jl} \varphi_{x_j} \varphi_{x_l} \left[\left(\frac{1}{\sqrt{M}}\right)'\right]^2 \to 0,$$

when $t \to 0$. Hence, we get the condition for symmetry

$$q(x) \geqslant -\text{Const. } e^{|x|^2} \,|\, x \,|^2 \log^2 |\, x \,|$$

for large $|\, x \,|$.

REMARK 1. If we set $M(t) \equiv 1$, the conditions for symmetry (2), (3) and (4) reduces to the less general but simpler form

$$\lim_{t \to 0} \int_t^{t_0} \frac{d\tau}{\sqrt{\rho(\tau)}} = \infty \tag{28}$$

$$-\frac{q(x)}{k(x)} \leqslant C_2 \left[\int_t^{t_0} \frac{d\tau}{\sqrt{\rho(\tau)}}\right]^2, \qquad C_2 > 0 \text{ constant.} \tag{29}$$

REMARK 2. In R_1, (1) with $b_j(x) = 0$ has the form

$$Au = \frac{1}{k(x)} \{- [p(x)\, u'(x)]' + q(x)\, u(x)\}. \tag{30}$$

We consider the case $G = (-\infty, \infty)$, and choose

$$\varphi(x) = \frac{1}{|x|}, \qquad M\left(\frac{1}{|x|}\right) = N(x)$$

and make a change of variable in the integrals. Then the criterion for symmetry can be formulated as follows:

THEOREM 2. *If there exists a positive function* $N(x) \in C^1$ $(0 < x_0 < x < \infty)$ *such that for sufficiently large* x

$$\lim_{x\to\infty} \int_{x_0}^{x} \sqrt{\frac{k(\xi)}{p(\xi)\, N(\xi)}}\, d\xi = \infty, \tag{31}$$

$$\frac{1}{N(x)} \leqslant C_1 \left[\int_{x_0}^{x} \sqrt{\frac{k(\xi)}{p(\xi)\, N(\xi)}}\, d\xi\right]^2, \tag{32}$$

$$\frac{1}{k(x)} \left\{\frac{2}{\epsilon}\, p(x) \left[\frac{d}{dx}\left(\frac{1}{\sqrt{N(x)}}\right)\right]^2 - \frac{q(x)}{N(x)}\right\} \leqslant C_2 \left[\int_{x_0}^{x} \frac{k(\xi)}{p(\xi)\, N(\xi)}\, d\xi\right]^2, \tag{33}$$

where $C_1 > 0$, $C_2 > 0$, $0 < \epsilon < 2$ *are constants, and equivalent conditions hold for* $x \to -\infty$, *then* A *in* $\mathfrak{A}$ *is symmetric.*

Theorem 2 is a generalization of the well-known Levinson criterion for the limit point case on both endpoints, which is a equivalent to symmetry of A in $\mathfrak{A}$. See [6] or [1].[1]

[1] One of the Levinson condition is

$$(*) \qquad \left| \sqrt{\frac{p(x)}{k(x)}}\, \frac{\dfrac{dN(x)}{dx}}{[N(x)]^{3/2}} \right| \leqslant \alpha, \qquad (\alpha \text{ constant}).$$

(*) implies

$$\left| \int_{x_0}^{x} \frac{\dfrac{dN(\xi)}{d\xi}}{N^2(\xi)}\, d\xi \right| \leqslant \int_{x_0}^{x} \left| \frac{\dfrac{dN(\xi)}{d\xi}}{N^2(\xi)} \right| d\xi = \int_{x_0}^{x} \sqrt{\frac{k(\xi)}{p(\xi) N(\xi)}} \left| \sqrt{\frac{p(\xi)}{k(\xi)}}\, \frac{\dfrac{dN(\xi)}{d\xi}}{N^{3/2}(\xi)} \right| d\xi$$

$$\leqslant \alpha \int_{x_0}^{x} \sqrt{\frac{k(\xi)}{p(\xi) N(\xi)}}\, d\xi$$

$$\left| \frac{1}{N(x)} - \frac{1}{N(x_0)} \right| \leqslant \alpha \int_{x_0}^{x} \sqrt{\frac{k(\xi)}{p(\xi) N(\xi)}}\, d\xi,$$

and hence with (31)

$$\frac{1}{N(x)} \leqslant \alpha \int_{x_0}^{x} \sqrt{\frac{k}{pN}}\, d\xi + \frac{1}{N(x_0)} \leqslant \alpha \left[\int_{x_0}^{x} \sqrt{\frac{k}{pN}}\, d\xi \right]^2$$

for sufficiently large x. Thus (32) is a generalization of (*).

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